Pre-class Warm-up!!!

Which of the following results have we already seen in a previous section, to do with a path c : [a,b] -> R^n and a vector field F on R^n ?

a. If F = grad f then $\int_c F \cdot d\underline{s}$ = f(c(b)) - f(c(a))

b. If $\int_c F \cdot d\underline{s}$ = f(c(b)) - f(c(a)) for some function f then F = grad f

c. If F = grad f then $\int_c F \cdot d\underline{s}$ does not depend on the particular choice of path from c(a) to c(b)

d. When n = 3, if F = grad f then curl F = 0

e. $\int_c F \cdot d\underline{s}$ does not depend on the particular choice of path from c(a) to c(b)

Quiz tomorrow on 7.1, 7.2, 8.1
Exam 2 Tuesday next week. Everything since Exam 1 up to 8.1.

a. Yes ✓ No

b. Yes No ✓ Not correct.

c. Yes ✓ No

d. Yes ✓ No

e. Yes No ✓ False

Section 8.3 Conservative vector fields

What we learn:

- a more complete angle on the gradient vector fields we have already studied
- path-independence
- another way to find a potential function whose gradient the field is
- a criterion for when a vector field is conservative
- slightly more elaborate applications, similar to what we have seen before

Theorem 7.
Let F be a vector field on R^n . The following are equivalent:

(ii) For any two oriented curves C_1 and C_2 that have the same end points $\int_{C_1} F \cdot d\underline{s} = \int_{C_2} F \cdot d\underline{s}$

(iii) F is the gradient of some function f

(i) For any oriented simple closed curve C ,

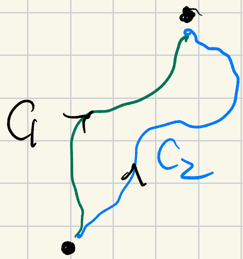
$$\int_C F \cdot d\underline{s} = 0$$

(iv) (assuming n = 3) curl F = 0

Definition: a vector field satisfying (ii) is called conservative

We have seen before?

- (ii) => (iii) Yes No
- (iii) => (ii) Yes No
- (i) => (ii) Yes No
- (ii) => (i) Yes No
- (iii) => (iv) Yes No
- (iv) => (iii) Yes No



Comments on:

(iii) F is the gradient of some function f

<=>

(iv) (assuming n = 3) curl F = 0

We have (sort of) seen (iii) $\Rightarrow$ (iv) before:

$$\nabla \times \nabla f = 0.$$

Check this: $\nabla \times \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$$= \left(\frac{\partial}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial y}, \quad , \quad \right)$$

Fact:

$\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y}$ if these derivatives exist and are continuous.

We get $\nabla \times \nabla f = 0$

$\text{curl} F = \nabla \times F$ $\quad \text{grad} f = \nabla f$ $\quad \text{div} F = \nabla \cdot F$

2-dimensional version

A vector field $F = (F_1, F_2)$ is ∇f for some $f: \mathbb{R}^2 \to \mathbb{R}$ $\Longleftrightarrow$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0 \iff \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}.$$

Recall always $\nabla \times \nabla f = 0$, $\nabla \cdot \nabla \times G = 0$

Theorem: When n = 3, given a vector field F we can write

F = curl G <=> Div F = 0

"$\Rightarrow$" if $F = \nabla \times G$ then $\nabla \cdot F = \nabla \cdot \nabla \times G = 0$

"$\Leftarrow$" is beyond our scope.

Questions (like 1 - 4, 17, 18):

Determine if the vector field
F(x,y,z) = (-2+4y, -4x, 0)
is a gradient vector field. If it is, find a function f so that F = grad f.
Determine whether F = curl G for some vector field G (but do not find G).

Solution. To see if $F = \nabla f$ for some f we calculate

$$\nabla \times F = \left(\frac{\partial}{\partial y} 0 - \frac{\partial}{\partial z}(-4x), \frac{\partial}{\partial z}(-2+4y) - \frac{\partial}{\partial x} 0, \frac{\partial}{\partial x}(-4x) - \frac{\partial}{\partial y}(-2+4y) \right)$$

$$= (0, 0, -4-4) = (0, 0, -8)$$

F is not ∇f for any f.

Next, we calculate $\nabla \cdot F$

$$= \frac{\partial}{\partial x}(-2+4y) + \frac{\partial}{\partial y}(-4x) + \frac{\partial}{\partial z} 0 = 0$$

Thus $F = \nabla \times G$ for some G.

Comments on:

(ii) For any two oriented curves C_1 and C_2 that have the same end points $\int_{C_1} F\cdot d\underline{s} = \int_{C_2} F\cdot d\underline{s}$

<=>

(iii) F is the gradient of some function f

We have seen (iii) $\Rightarrow$ (ii). If $F = \nabla f$ then

$$\int_{\text{any } C} F = f(c(b)) - f(c(a))$$

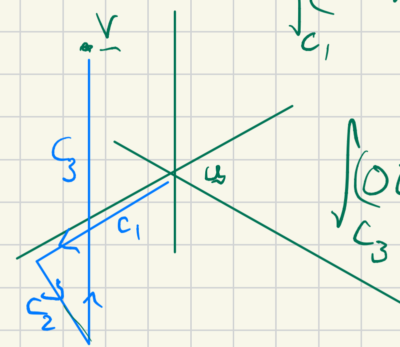
is independent of choice of c.

(ii) $\Rightarrow$ (iii) Assume (ii). We construct a function f with $F = \nabla f$. Pick a point $\underline{u}$ in $\mathbb{R}^n$. For any vector $\underline{v}$ define $f(\underline{v}) = \int_C F\cdot d\underline{s}$ where C is any path from $\underline{u}$ to $\underline{v}$.

We have to show $\nabla f = F$.

Consider $F_3 \overset{?}{=} \frac{\partial}{\partial z} f$

$$\int_{C_1} (0\ 0\ F_3)\cdot d\underline{s} = \int_{C_2} (0\ 0\ F_3)\cdot d\underline{s} = 0.$$



$$\int_{C_3} (0\ 0\ F_3)\cdot d\underline{s} = \int_{C_3} F_3\, dz$$

is an antiderivative of F_3, $\frac{\partial f}{\partial z} = F_3$.

Do the same with components F_1 and F_2, and add.

We have a new way to compute a potential function f for F when F is conservative, but in practice it is not an improvement on the way we have already seen.

Example: Find f so that F = grad f when F(x,y,z) = (e^x sin y, e^x cos y, z^2).

Solution: We solve

$$\frac{\partial f}{\partial x} = e^x \sin y \quad \frac{\partial f}{\partial y} = e^x \cos y \quad \frac{\partial f}{\partial z} = z^2$$

$$f = e^x \sin y + \frac{1}{3} z^3$$

Example: Find $\int_c \underline{F} \cdot d\underline{s}$

when c(t) = (t, e^(sin t)), 0 ≤ t ≤ π) and F(x,y,z) = (y, x)

Solution.

In fact F is conservative

$F = \nabla f$ where $f(x,y) = xy$

The integral is $f(c(\pi)) - f(c(0))$

$= \pi \cdot 1 - 0 \cdot 1 = \pi$.

Comments on

(ii) For any two oriented curves C_1 and C_2 that have the same end points

$$\int_{C_1} F \cdot d\underline{s} = \int_{C_2} F \cdot d\underline{s}$$

<=>

(i) For any oriented simple closed curve C ,

$$\int_C F \cdot ds = 0$$

(ii) $\Rightarrow$ (i). Let C start and finish at $\underline{u}$. Let C_1 be the constant curve $c_1(t) = \underline{u}$ for all t

$$\int_C F \cdot d\underline{s} = \int_{C_1} F \cdot d\underline{s}$$

$c_1'(t) = 0$ so $\int_{C_1} F \cdot d\underline{s} = 0$,

hence $\int_C F \cdot d\underline{s} = 0$.